MACMAHON'S SUM-OF-DIVISORS FUNCTIONS, CHEBYSHEV POLYNOMIALS, AND QUASI-MODULAR FORMS

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Abstract. We investigate a relationship between MacMahon's generalized sum-of-divisors functions and Chebyshev polynomials of the first kind. This determines a recurrence relation to compute these functions, as well as proving a conjecture of MacMahon about their general form by relating them to quasi-modular forms. These functions arise as solutions to a curve-counting problem on Abelian surfaces.

1. Introduction

The sum-of-divisors function $\sigma_k(n)$ is defined to be

$$\sigma_k(n) = \sum_{d|n} d^k.$$

For $k = 1$, this has as a generating function

$$A_1(q) = \sum_{k=1}^{\infty} \sigma_1(n) q^n = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2}.$$

As a generalization of this notion, MacMahon introduces in the paper [5, pp. 303, 309] the generating functions

$$A_k = \sum_{0<m_1<\cdots<m_k} \frac{q^{m_1+\cdots+m_k}}{(1-q^{m_1})^2\cdots(1-q^{m_k})^2}$$

$$C_k = \sum_{0<m_1<\cdots<m_k} \frac{q^{2m_1+\cdots+2m_k-k}}{(1-q^{2m_1-1})^2\cdots(1-q^{2m_k-1})^2}.$$

These provide generalizations in the following sense.

Fix a positive integer k. We define $a_{n,k}$ to be the sum

$$a_{n,k} = \sum s_1 \cdots s_k$$

where the sum is taken over all possible ways of writing $n = s_1 m_1 + \cdots + s_k m_k$ with $0 < m_1 < \cdots < m_k$. Note that for $k = 1$ this is nothing but $\sigma_1(n)$, the usual sum-of-divisors function. It can then be shown that we have

$$A_k(q) = \sum_{n=1}^{\infty} a_{n,k} q^n.$$

Similarly, we define $c_{n,k}$ to be

$$c_{n,k} = \sum_{1} s_1 \cdots s_k$$

where the sum is over all partitions of n into

$$n = s_1(2m_1 - 1) + \cdots + s_k(2m_k - 1)$$

with, as before $0 < m_1 < \cdots < m_k$. For $k = 1$ this is the sum over all divisors whose conjugate is an odd number. As for the case of $a_{n,k}$, we have

$$C_k(q) = \sum_{n=1}^{\infty} c_{n,k} q^n.$$

We recall also that Chebyshev polynomials [1, p. 101] are defined via the relation

$$T_n(\cos\theta) = \cos(n\theta).$$

With these we form the following generating functions.

$$F(x, q) := 2\sum_{n=0}^{\infty} T_{2n+1}(\tfrac{1}{2}x) q^{n^2+n}$$

$$G(x, q) := 1 + 2\sum_{n=1}^{\infty} T_{2n}(\tfrac{1}{2}x) q^{n^2}.$$

The results of this paper are the following.

Theorem 1. *We have the following equalities:*

$$F(x, q) = (q^2; q^2)^3_\infty \sum_{k=0}^{\infty} A_k(q^2) x^{2k+1}$$

$$G(x, q) = \frac{(q; q)_\infty}{(-q; q)_\infty} \sum_{k=0}^{\infty} C_k(q) x^{2k}$$

where $(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$.

Corollary 2. *The functions $A_k(q)$ and $C_k(q)$ can be written as*

$$A_k(q) = \frac{(-1)^k}{(2k+1)!(q; q)^3_\infty} \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{(n+k)!}{(n-k)!} q^{\frac{1}{2}n(n+1)}$$

$$C_k(q) = \frac{(-1)^k(-q; q)^\infty}{(2k)!(q; q)_\infty} \sum_{n=1}^{\infty} (-1)^n 2n \frac{(n+k-1)!}{(n-k)!} q^{n^2}.$$

Corollary 3. *The functions A_k and C_k satisfy the recurrence relations*

$$A_k(q) = \frac{1}{(2k+1)2k}\Big(\big(6A_1(q) + k(k-1)\big)A_{k-1}(q) - 2q\frac{d}{dq}A_{k-1}(q)\Big)$$

$$C_k(q) = \frac{1}{2k(2k-1)}\Big(\big(2C_1(q) + (k-1)^2\big)C_{k-1}(q) - q\frac{d}{dq}C_{k-1}(q)\Big).$$

Our final result settles a long-standing conjecture of MacMahon. In MacMahon's paper [5, p. 328] he makes the claim

> The function $A_k = \sum a_{n,k} q^n$ has apparently the property that the coefficient $a_{n,k}$ is expressible as a linear function of the sum of the uneven powers of the divisors of n. I have not succeeded in reaching the general theory...

What we prove is the following.

Corollary 4. *The functions $A_k(q)$ are in the ring of quasi-modular forms.*

We will also discuss in section 3 some applications of this result to an enumerative problem involving counting curves on abelian surfaces.

2. Proofs

Proof of theorem 1. Beginning with the series $F(x,q)$, and letting $x = 2\cos\theta$ we find

$$\begin{aligned} F(x,q) &= 2\sum_{n=0}^{\infty} T_{2n+1}(\cos\theta)q^{n^2+n} \\ &= 2\sum_{n=0}^{\infty} \cos\big((2n+1)\theta\big)q^{n^2+n} \\ &= \sum_{n=0}^{\infty}\Big(e^{i(2n+1)\theta} + e^{-i(2n+1)\theta}\Big)q^{n^2+n} \\ &= \sum_{n=0}^{\infty} e^{i(2n+1)\theta}e^{n^2+n} + \sum_{n=0}^{\infty} e^{-i2n+1\theta}q^{n^2+n} \end{aligned}$$

where in the latter sum, letting $n \mapsto -n-1$ we obtain

$$F(x,q) = e^{i\theta}\sum_{n=-\infty}^{\infty} e^{2ni\theta}q^{n^2+n}.$$

Using the Jacobi triple product [1, p. 497, Thm 10.4.1] we see that this is equal to

$$\begin{aligned} e^{i\theta}\sum_{n=-\infty}^{\infty} e^{2ni\theta}q^{n^2+n} &= e^{i\theta}(-e^{-2i\theta};q^2)_\infty(-q^2e^{2i\theta};q^2)_\infty(q^2;q^2)_\infty \\ &= \underbrace{(e^{i\theta}+e^{-i\theta})}_{x}(q^2;q^2)_\infty\prod_{m=1}^{\infty}\big(q+\underbrace{2\cos(2\theta)}_{x^2-2}q^{2m}+q^{4m}\big) \\ &= x(q^2;q^2)_\infty\prod_{m=1}^{\infty}\big((1-q^{2m})^2+x^2q^{2m}\big) \\ &= x(q^2;q^2)_\infty^3\prod_{m=1}^{\infty}\Big(1+x^2\frac{q^{2m}}{(1-q^{2m})^2}\Big) \\ &= (q^2;q^2)_\infty^3\sum_{k=0}^{\infty}A_k(q^2)x^{2k+1} \end{aligned}$$

and thus comparing coefficients of x^{2k+1} yeilds the result.

We ply a similar trick for $G(x,k)$. In that case we have

$$\begin{aligned} G(x,q) &= 1+2\sum_{n>0}T_{2n}(\cos\theta)q^{n^2} \\ &= 1+2\sum_{n>0}\cos(2n\theta)q^{n^2} \\ &= \sum_{n=-\infty}^{\infty} e^{2ni\theta}q^{n^2} \end{aligned}$$

which, again, by the Jacobi triple product yields

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} e^{2ni\theta} q^{n^2} &= (q^2;q^2)_\infty(-qe^{2i\theta};q^2)_\infty(-qe^{-2i\theta};q^2)_\infty \\
&= (q^2;q^2)_\infty \prod_{m=1}^{\infty} \big(1+\underbrace{2\cos(2\theta)}_{x^2-2} q^{2m-1}+q^{4m-2}\big) \\
&= (q^2;q^2)_\infty \prod_{m=1}^{\infty} \big((1-q^{2m-1})^2 + x^2 q^{2n-1}\big) \\
&= \underbrace{(q^2;q^2)_\infty (q;q^2)_\infty^2}_{\frac{(q;q)_\infty}{(-q;q)_\infty}} \prod_{m=1}^{\infty} \Big(1+x^2\frac{q^{2m-1}}{(1-q^{2m-1})^2}\Big) \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty} \sum_{k=0}^{\infty} C_k(q) x^{2k}
\end{aligned}$$

which completes the theorem. □

To deduce Corollary 2, we begin by expanding the series $F(x,q)$ (and similarly, $G(x,q)$) in powers of x, i.e.

$$\begin{aligned}
F(x,q) &= xf_0(q) + x^3 f_1(q) + x^5 f_2(q) + \cdots + x^{2k+1} f_k(q) + \cdots \\
G(x,q) &= g_0(q) + x^2 g_1(q) + x^4 g_2(q) + \cdots + x^{2k} g_k(q) + \cdots .
\end{aligned}$$

Now, it can be shown that the coefficients of x^{2k} in $2T_{2n}(\frac{1}{2}x)$ and of x^{2k+1} in $2T_{2n+1}(\frac{1}{2}x)$ are respectively given by

$$2n(-1)^{n-k}\frac{(n+k-1)!}{(n-k)!(2k)!} \qquad (-1)^{n-k}(2n+1)\frac{(n+k)!}{(n-k)!(2k+1)!}$$

and thus we have

$$\begin{aligned}
f_k(q) &= \frac{(-1)^k}{(2k+1)!}\sum_{n=0}^{\infty}(-1)^n(2n+1)\frac{(n+k)!}{(n-k)!}q^{n^2+n} \\
g_k(q) &= \frac{(-1)^k}{(2k)!}2\sum_{n=1}^{\infty}(-1)^n n\frac{(n+k-1)!}{(n-k)!}q^{n^2}.
\end{aligned}$$

As theorem 1 implies that $f_k(q) = (q^2;q^2)_\infty^3 A_k(q^2)$ and $g_k(q) = \frac{(q;q)_\infty}{(-q;q)_\infty} C_k(q)$, we see that Corollary 2 follows.

Next, letting

$$\begin{aligned}
f_0(q) &= \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n} = (q^2;q^2)_\infty^3 \\
g_0(q) &= 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}
\end{aligned}$$

and defining the operators $D_\ell = q\frac{d}{dq} - \ell(\ell-1)$ and $D'_\ell = q\frac{d}{dq} - (\ell-1)^2$, we then have that

$$f_k(q) = \frac{(-1)^k}{(2k+1)!} D_k \cdots D_1 f_0(q)$$
$$g_k(q) = \frac{(-1)^k}{(2k)!} D'_k \cdots D'_1 g_0(q).$$

From these formulae we note that the functions f_k, g_k satisfy the recursion relations

$$f_k(q) = \frac{-1}{(2k+1)2k}\Big(q\frac{d}{dq} - k(k-1)\Big) f_{k-1}(q)$$
$$g_k(q) = \frac{-1}{2k(2k-1)}\Big(q\frac{d}{dq} - (k-1)^2\Big) g_{k-1}(q).$$

Noting again that $f_k(q) = (q^2;q^2)_\infty^3 A_k(q^2)$ and $g_k(q) = \frac{(q;q)_\infty}{(-q;q)_\infty} C_k(q)$, we now obtain the recurrence relation of Corollary 3 between the functions $A_k(q)$ and $C_k(q)$.

Our final result requires a bit of explanation. It is well known that the ring of modular forms for the full modular group $\Gamma = PSL_2(\mathbb{Z})$ is the polynomial ring in the generators E_4, E_6, where

$$E_{2k}(q) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

are the Eisenstein series of weights $2k$. There are no modular forms of weight 2 for Γ, but $E_2 = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n$ is a *quasi*-modular form (See [4]).

The relevant fact for this paper is that the ring of all such objects (which contains the ring of modular forms as a subring) is the ring generated either by E_2, E_4, and E_6, or by $q\frac{d}{dq}$ and by E_2. Noting then that $A_1(q) = \frac{1-E_2(q)}{24}$, the recurrence relation from Corollary 3 implies that each $A_k(q)$ lies in this ring, and hence the conclusion follows.

3. Applications

The functions $A_k(q)$ and $C_k(q)$ arise naturally in the following problem in enumerative algebraic geometry.

Let $A \subset \mathbb{P}^N$ be a generic polarized abelian surface. There are then a finite number of hyperplane sections which are hyperelliptic curves of geometric genus g and having $\delta = N - g + 2$ nodes. The number of such curves, $N(g,\delta)$ is independent of the choice of A and these numbers can be assembled into a generating function

$$F(x,u) = \sum_{g,\delta} N(g,\delta) x^g u^\delta.$$

The coefficient of x^g in F is given by a certain homogeneous polynomial of degree $g-1$ in the functions $A_k(u^4)$ and $C_k(u^2)$.

This formula is derived by relating hyperelliptic curves on A to genus zero curves on the Kummer surface $A/\pm 1$. The latter is computed using orbifold Gromov-Witten theory, the Crepant resolution conjecute [2] and the Yau-Zaslow formula [6] [3]. This will be described further in the second author's thesis.

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